B-Spline Interpolation

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Overview

- Introduction
  - Piece-wise curve
  - Parametric representation of curves
- Curves
  - Cubic curve
  - Bezier curve
  - B-Spline curve
- Application

Piece-wise curve

- Linear interpolation: curve is approximated by line segments
- Piece-wise polynomial curve: the curve is approximated by segments of parametric polynomial curve

Parametric representation of curves

- 2 functions \( x = g(t) \), \( y = h(t) \) where \( t \) varies in the interval \([0,1]\) defines a curve in 2D
  - Ex: \( y = f(x) \)
  - \((x,y) \rightarrow (g(t), h(t))\)
- If \( x = g(t) \), \( y = h(t) \) are polynomials of degree 3, we get a cubic curve
- Reasons why cubic curve are most often used: polynomials of degree 2 or less don’t satisfy \( C^2 \) continuity, while higher degree tend to complicate controls

Cubic curve

- A segment of 3D curve is given by three polynomials of degree 3
  - \( X(t) = a_1t^3 + b_1t^2 + c_1t + d_1 \)
  - \( Y(t) = a_2t^3 + b_2t^2 + c_2t + d_2 \)
  - \( Z(t) = a_3t^3 + b_3t^2 + c_3t + d_3 \)
- Not having global control property

Bezier curve

- Bezier curve employs a set of ordered points \( b_0, b_1, \ldots, b_n \) to approximate the curve
- The set of ordered points \( b_0, b_1, \ldots, b_n \) is known as Control Points or Control Vertices
- The polygon constructed with \( b_0, b_1, \ldots, b_n \) is known as Control Polygon
- The shape of a Bezier curve generally follows the shape of control polygon
A Bezier curve lies within a convex hull of the control polygon.

Convex Hull – the minimum convex polygon enclosing a set of given points.

Mathematically, a Bezier curve is defined as:

\[ p(t) = \sum_{i=0}^{n} b_i J_{n,i}(t) \quad 0 \leq t \leq 1 \]

where \[ J_{n,i}(t) = \binom{n}{i} (-t)^i (1-t)^{n-i} \] and \[ \binom{n}{i} = \frac{n!}{i!(n-i)!} \]

- Bezier curve for degree 3

\[ P(t) = (1-t)^3 b_0 + 3(1-t)^2 b_1 + 3(1-t) b_2 + t^3 b_3 \]

- Limitations of Bezier curve
  - Not having local control property

B-spline curve

- Cubic B-spline
- Cubic segment \( p_i(t) \) is defined by 4 control points \( P_{i-1}, P_i, P_{i+1}, P_{i+2} \)

\[ p_i(t) = (t^3, t^2, t, 1) M \]

\[ M : 4 \times 4 \text{ basis matrix} \]

Matrix Expression of Cubic B-Spline Curve

\[ (t^3, t^2, t, 1) M = (a_1, b_1, c_1, d_1) \]

\[ \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} \]

\[ = (a t^3 + a_d t^3 + a_0 b t^2 + b_0 t^2 + b_0 t + b_0, c_1 t^3 + c_1 t^3 + c_1 c_1 t^2 + c_1 c_1 t^2 + c_1 c_1 t + c_1 c_1) \]

Four constrains:

\[ p_i(t) = p_i(0) \]
\[ p_i'(t) = p_i'(0) \]
\[ p_i''(t) = p_i''(0) \]
\[ a(t) + b(t) + c(t) + d(t) = 1 \]

2D Cubic B-Spline Function

\[ T(x,y) = \sum_{n=0}^{3} \sum_{u=0}^{3} B_n(u) B_s(v) \Phi_{n,m,j+i} \]

\[ i = \frac{x}{n_x} - 1, \quad j = \frac{y}{n_y} - 1 \]

\[ u = \frac{x}{n_u}, \quad v = \frac{y}{n_v} \]

\[ B_n(u) = (1-u)^3 / 6 \]
\[ B_i(u) = (3u^3 - 6u^2 + 4) / 6 \]
\[ B_s(v) = (3v^3 - 3v^2 + 3v + 1) / 6 \]
\[ B_i(u) = u^3 / 6 \]